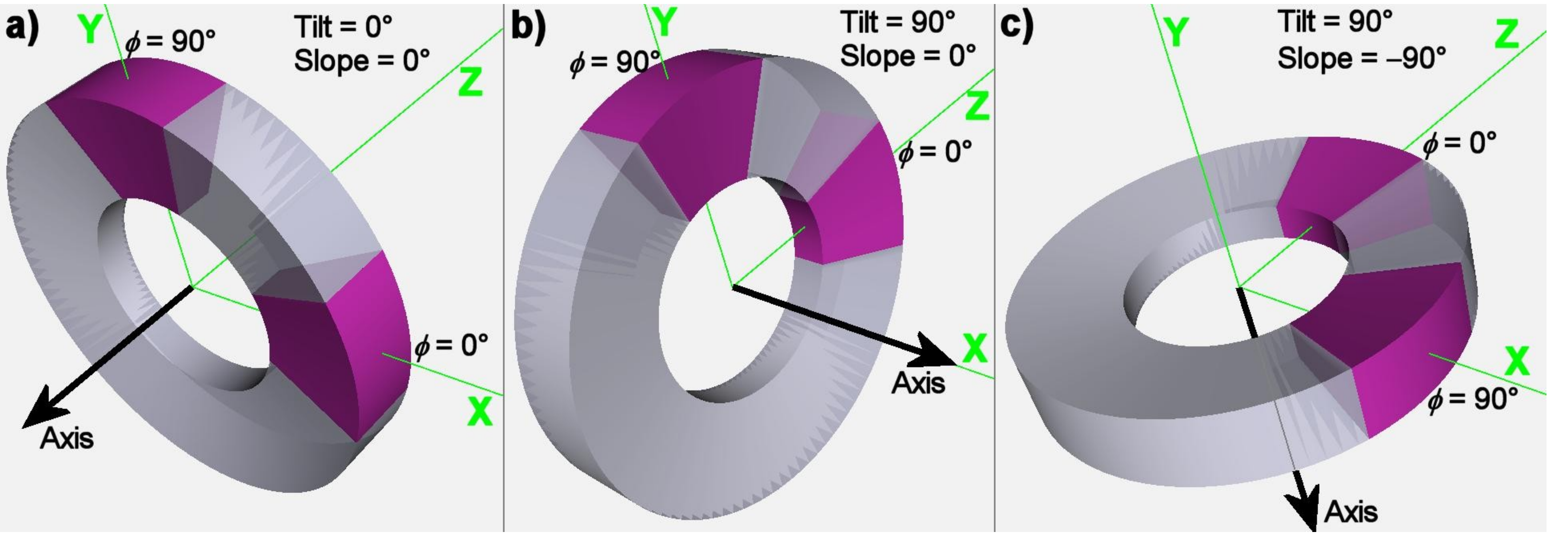
Coordinate transforms for Radial Collimator / corrected December 12, 2009



a) The coordinate system in which the intrinsic parameters are defined. The default Axis of the donut is the –Z axis so that the azimuthal direction is *right-handed* about the Axis. On the other hand (so to speak), NISP coordinates are left-handed. In either case, the azimuthal direction is measured from the +X axis to the +Y axis. Two collimator segments are shown, centered at $\phi = 0°$ and at $\phi = 90°$. The blades in the collimator are always parallel to the Axis of the donut. The collimation is in the azimuthal direction.

b) Apply **Tilt**, by an angle T about the +Y axis. The general rotation matrix is

$$\mathbf{T} = \begin{pmatrix} \cos T & 0 & -\sin T \\ 0 & 1 & 0 \\ \sin T & 0 & \cos T \end{pmatrix}$$

For the case shown, $T = 90°$, the Axis is now in the +X direction and collimation is in the polar direction around the meridian. It is expected that radial collimators will be either azimuthal ($T = 0°$) or polar ($T = 90°$), but the rotation matrix is general.

c) Apply **Slope**, by an angle S upward from the X-Z plane. The Slope is about the X axis *before* Tilt is applied. The sign of S is determined by the direction the *Axis* rotates, and is *opposite from other elements in NISP*. The right-handed rotation matrix is

$$\mathbf{S} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos S & -\sin S \\ 0 & \sin S & \cos S \end{pmatrix}$$

With $S = -90°$ as shown, the Axis is downward and collimation is in the polar direction, in the X-Z plane. Note that in this particular orientation with $T = 90°$, the angle ϕ is increasing from the +Z axis toward the +X axis. This is the "World" coordinate system in which neutrons are transported.

To find the equation of a surface in the world coordinate of c), we must transform points (x,y,z) back to the intrinsic system a) and insert the transformed (x',y',z') into the surface equations there. The back-transformation is

$$(\mathbf{T}\times\mathbf{S})^{-1}=\mathbf{S}^{-1}\times\mathbf{T}^{-1}=\begin{pmatrix}\cos T & 0 & \sin T\\ -\sin T\sin S & \cos S & \cos T\sin S\\ -\sin T\cos S & -\sin S & \cos T\cos S\end{pmatrix}$$

The determinant of the matrix is 1, and the inverse is equal to the transpose. Including translation to an origin at (*X*,*Y*,*Z*) in the world system, the intrinsic coordinates are

$$\begin{pmatrix}x'\\ y'\\ z'\end{pmatrix}=\begin{pmatrix}\cos T & 0 & \sin T\\ -\sin T\sin S & \cos S & \cos T\sin S\\ -\sin T\cos S & -\sin S & \cos T\cos S\end{pmatrix}\begin{pmatrix}x-X\\ y-Y\\ z-Z\end{pmatrix}$$

These relations are all that is needed to define *plane* surfaces, but for quadratic surfaces we need the squares of the transformed coordinates. In both the linear and quadratic expressions, it is convenient to use upper-case Greek letters (Chi,Ypsilon,Zeta) for the respective expressions for $(\alpha X+\beta Y+\gamma Z)$. For this transformation matrix,

$$\begin{pmatrix}\mathbf{X}\\ \mathbf{Y}\\ \mathbf{Z}\end{pmatrix}=\begin{pmatrix}\cos T & 0 & \sin T\\ -\sin T\sin S & \cos S & \cos T\sin S\\ -\sin T\cos S & -\sin S & \cos T\cos S\end{pmatrix}\begin{pmatrix}X\\ Y\\ Z\end{pmatrix}$$

Consider the form

$$x' = \alpha\, x+\beta\, y+\gamma\, z-\mathbf{X}$$

Squaring and separating out squared, first-power, and cross terms,

$$\begin{aligned}x'^2 = {} & \alpha^2 x^2+\beta^2 y^2+\gamma^2 z^2+\mathbf{X}^2\\ & -2\alpha\,\mathbf{X}\,x-2\beta\,\mathbf{X}\,y-2\gamma\,\mathbf{X}\,z\\ & +2\,\alpha\,\beta\,(x\,y)+2\,\beta\gamma\,(y\,z)+2\,\gamma\,\alpha\,(z\,y)\end{aligned}$$

Any quadratic surface has 10 coefficients in the world coordinate system:

$$A x^2+B x+C y^2+D y+E z^2+F z+G+P x y+Q y z+R z x\ =\ 0$$

Surfaces 1 and 2 are intrinsic cylinders of the form $x'^2+y'^2=r^2$. The sum gives

$$A=\ \cos^2 T+\sin^2 T\sin^2 S$$

$$B=-2\,\mathbf{X}\cos T+2\,\mathbf{Y}\sin T\sin S$$

$$C=\ \cos^2 S$$

$$D=-2\,\mathbf{Y}\cos S$$

$$E=\ \sin^2 T+\cos^2 T\sin^2 S$$

$$F=-2\,\mathbf{X}\sin T-2\,\mathbf{Y}\cos T\sin S$$

$$G=\mathbf{X}^2+\mathbf{Y}^2-r^2$$

$$P=-2\,\sin T\sin S\cos S\qquad =\ -\sin T\sin 2S$$

$$Q=\ 2\,\cos T\sin s\cos S\qquad =\ \ \cos T\sin 2S$$

$$R=\ 2\,\sin T\cos T-2\sin T\cos T\sin^2 S\ =\ \ \sin 2T\cos^2 S$$

The use of double-angles somewhat simplifies calculation of the cross terms.

Intrinsic surfaces 3 and 4 are the “back” and the “front” of the donut in Fig. 1a. Given the thicknesses *h* and *H* at radii *r* and *R*, the slopes of the two surfaces are

$$m=\pm\left(\tfrac{1}{2}\right)(H-h)/(R-r)$$

and the respective intercepts are found by extrapolating to the Z'-axis:

$$z_0 = \pm (h/2 - m\,r)\,.$$

Ideally these surfaces would be cones, and the equation in intrinsic coordinates would be:

$$z' = m\sqrt{x'^2 + y'^2} + z_0$$

$$m^2(x'^2 + y'^2) - z'^2 + 2\,z_0\,z' - z_0{}^2 = 0$$

The signs are chosen such that the region of interest is on the exterior (+) side of both cones. The problem is that NISP can not distinguish between the two sheets of the cone. If $z_0 \neq 0$, there will be a minimum interior radius where sheets from the two surfaces intersect. Therefore cones can only be implemented if $z_0 = 0$, in which case the two surfaces are the two sheets of the *same* cone, that is, the front and back surfaces converge to the origin. In that case, the world coefficients for the cone are

$$A = m^2(\cos^2 T + \sin^2 T \sin^2 S) - \sin^2 T \cos^2 S$$

$$B = m^2(-2\,\mathbf{X}\cos T + 2\,\mathbf{Y}\sin T \sin S) - 2\,\mathbf{Z}\sin T \cos S$$

$$C = m^2(\cos^2 S) - \sin^2 S$$

$$D = m^2(-2\,\mathbf{Y}\cos S) + 2\,\mathbf{Z}\sin S$$

$$E = m^2(\sin^2 T + \cos^2 T \sin^2 S) - \cos^2 T \cos^2 S$$

$$F = m^2(-2\,\mathbf{X}\sin T - 2\,\mathbf{Y}\cos T \sin S) - 2\,\mathbf{Z}\cos T \cos S$$

$$G = m^2(\mathbf{X}^2 + \mathbf{Y}^2) - \mathbf{Z}^2$$

$$P = (m^2 + 1)(-\sin T \sin 2S)$$

$$Q = (m^2 + 1)\cos T \sin 2S$$

$$R = (m^2 + 1)\sin 2T \cos^2 S$$

If $z_0 \neq 0$, surfaces 3 and 4 will be planes tangent at ϕ, the median azimuthal angle of the region. In intrinsic coordinates, the region between the planes is

$$z_0 + m\,(\cos\phi\; x' + \sin\phi\; y') \;\;\geq\;\; z' \;\;\geq\;\; -z_0 - m\,(\cos\phi\; x' + \sin\phi\; y')$$

$$m\,(\cos\phi\; x' + \sin\phi\; y') \mp z' + z_0 \;\geq\; 0$$

where $-z'$ is the back surface and $+z'$ the front, and the region is on the + side of both. Then $\{A, C, E, P, Q, R\} = 0$, and

$$B = \;m\cos\phi\cos T - m\sin\phi\sin T\sin S \;\pm \sin T\cos S$$

$$D = \;m\sin\phi\cos S \;\pm \sin S$$

$$F = \;m\cos\phi\sin T + m\sin\phi\cos T\sin S \;\mp \cos T\cos S$$

$$G = -m\cos\phi\,\mathbf{X} \;\; - m\sin\phi\,\mathbf{Y} \;\; \pm\; \mathbf{Z} \;\; + z_0$$

The remaining surfaces are radial planes that contain the donut Axis, inclined at various azimuthal angles ϕ in the intrinsic system. Choosing signs such that the side with larger ϕ is +, the form is

$$-\sin\phi\; x' + \cos\phi\; y' = 0$$

$$B = -\sin\phi\cos T - \cos\phi\sin T\sin S$$

$$D = \;\cos\phi\cos S$$

$$F = -\sin\phi\sin T + \cos\phi\cos T\sin S$$

$$G = \;\sin\phi\,\mathbf{X} \;\; - \cos\phi\,\mathbf{Y}$$

The following picture is a model of HIPPO (with the TAP-98 high-pressure press installed), including eight modules of radial collimators. All are centered at $\phi = 90°$, and all have tilt angle $T = 90°$. Each module has a different slope angle, S, to match the detector locations of HIPPO. They are $S = \pm 56.4°$, ±83.9°, ±111.3°, and ±138.7°. The eight detector panels at 90° are shown in green. The radial collimators reduce the number of neutrons reaching the detector after scattering from the iron of the press and the diamond anvils of the pressure cell.